Finite-time enclosing control for multiple moving targets with control input saturation

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***Abstract*—Enclosing formation has been shown very popular in nature swarms and therefore has been widely applied for target entrapping and protecting in the artificial swarms. This paper addresses the finite-time enclosing control problem using multi-agent system (MAS) for multiple moving targets with control input saturation. It is guaranteed that both the including angle and relative distance can converge to the desired formation in finite time. Specifically, we focus on the case where the enclosing agents are limited by the control input saturation. Two novel finite-time sliding modes have been proposed to compensate for the saturation by monitoring the difference between the desired control command and the executed control command. On this basis, the finite time controllers are designed to achieve the desired enclosing formation. Finally, simulation results validate the correctness and effectiveness of the proposed methods.**

***Index Terms*—Enclosing control, Multi-agent system, Control saturation, Finite-time convergence, Sliding mode control, Including angle.**

I. Introduction

RCENT years have witnessed the great development of cooperative control for the MAS [1], [2]. Among those techniques, the enclosing control is becoming more and more popular due to its broad potential in applications such as entrapping or protecting important unit in contest environment, search and rescue, surveillance and so on. Enclosing control problem involves applying a set of mobile agents to achieve a predefined formation such that single or multiple targets can always be contained. In order to provide full capacity of enclose, the formation is usually designed to rotate or encircle around these targets.

Early work mainly focus on single target case, due to its advantage of determining the center and radius of the desired enclosing formation. In such a case, the followers can simply fix the rotating center on the target and control their relative distance to the desired radius according to its observation to the target. As a result, a variety of control strategies have been proposed under different scenarios. [3] has proposed a distributed and communication-free enclosing strategy in 3D space, called the cyclic pursuit strategy where each follower simply pursuits its predecessor. [4] considers the moving target whose velocity is piece-wisely constant with dynamically changing network topology. Enclosing problem using bearing-only measurement has been investigated in [5], where the target is assumed to be static and it is not necessary to communicate among followers network. This method is soon extended in [6] to the moving target case using bearing-only measurement in the expense of the introduction of communication to mutually achieve the consensus of estimates of target velocity. In addition, the enclosing problem for nonholonomic followers and moving target is studied in [7].

Enclosing multiple targets is much more difficult than the single target case. The difficulties are mainly limit of observations between followers and targets, which results in the necessity to communicate with other followers to share their observation for the estimation of rotating center and determine the radius. For static targets, both first-order and second-order followers have been discussed in [8] and [9] respectively. Moving on, several finite-time estimators for the target's geometrical center has been proposed in [10]–[13]. In addition to the aforementioned research, the followers also suffer from control input saturation in real platform. The enclosing problem of Euler–Lagrange systems with input constraints has been investigated in [14]. A two-stage enclosing problem for linear MAS subject to input saturation has been considered in [15].

This paper addresses the finite-time enclosing problem for multiple moving targets with control input saturation. Two novel finite-time sliding models have been designed to compensate the control input saturation. Based on the sliding models, the controllers are then designed such that the desired enclosing formation can be achieved in finite time.

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II. Preliminaries

Before presenting the main results, the finite-time stability theorem [16] is important.

Theorem 1. *Consider the nonlinear system* $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$ *with* $f(\mathbf{0}) = \mathbf{0}$. *Suppose that there is a* C^1 *function* $V(\boldsymbol{x}_0)$ *defined on a neighborhood of the origin and real numbers* $c > 0, \alpha \in (0,1)$, *such that*

$$\dot{V}(\boldsymbol{x}) \leq -cV^{\alpha}(\boldsymbol{x}),$$

then the origin of system is finite-time stable and the upper bound of the settling time is

$$T \leq \frac{V^{1-\alpha}(\mathbf{0})}{c(1-\alpha)}.$$

In addition, given an real constant $\gamma < V(\mathbf{0})$, *then the settling time* T_1 *guaranteeing* $\boldsymbol{t} < \gamma, \forall t > T_1$ *is*

$$T_1 \leq \frac{V^{1-\alpha}(\mathbf{0}) - \gamma^{1-\alpha}}{c(1-\alpha)}.$$

Besides, another methods for proving finite-time convergence of a closed-loop system is by applying the homogeneous theorem. Firstly, the degree of homogeneity is defined as follows:

Definition 1. *A function* $V : \mathbb{R}^n \to \mathbb{R}$ *is homogeneous of degree* l *with respect to the "standard dilation":*

$$\Delta_\lambda(x_1, x_2, \cdots, x_n) = (\lambda x_1, \lambda x_2, \cdots, \lambda x_n)$$

if and only if:

$$V(\lambda x_1, \lambda x_2, \cdots, \lambda x_n) = \lambda^l V(x_1, x_2, \cdots, x_n)$$

Definition 2. *Considering the following system* $\dot{\boldsymbol{x}} = f(\boldsymbol{x}), \boldsymbol{x} \in \mathbb{R}^n$, *it is called to be homogeneous of degree* q *with respect to the standard dilation if and only if the* i*-th component* f_i *is homogeneous of degree* $q+1$ *with respect to the standard dilation, i.e.* $f_i(\lambda x_1, \lambda x_2, \cdots, \lambda x_n) = \lambda^{q+1} f_i(x_1, \cdots, x_n)$, $\lambda > 0, i = \{1, 2, \cdots, n\}$.

Therefore, the following algorithm [16] can also be used for proving the finite-time convergence of a system.

Theorem 2. *Let* $f(\boldsymbol{x}), \boldsymbol{x} \in \mathbb{R}^n$ *be a homogeneous vector field of degree* q *with respect to the standard dilation, then the nonlinear system* $\dot{\boldsymbol{x}} = f(\boldsymbol{x}), \boldsymbol{x} \in \mathbb{R}^n$ *is finite-time stable if and only if it is asymptotically stable and* $q < 0$.

Besides, the following lemma is instrumental for future analysis,

Lemma 1. *Given a vector* $\boldsymbol{x} \in \mathbb{R}^n$ *and a positive* $0 < p < 1$, *then we have* $\left(\sum_{i=1}^n |x_i|\right)^p \leq \sum_{i=1}^n |x_i|^p \leq n^{1-p}\left(\sum_{i=1}^n |x_i|\right)^p$

III. Problem Formulation

Consider the case where a group of mobile agents (called the followers), moving on the 2D plane, are deployed to entrap or protect another group of agents (called the leaders). The objective for followers is to form a predefined enclosing formation using only local observation to the leaders and the exchanged messages in the followers' network such that the leaders will always stay within the convex hull formed by followers. The dynamics of leaders and followers are all assumed to be point-mass model, i.e. ,

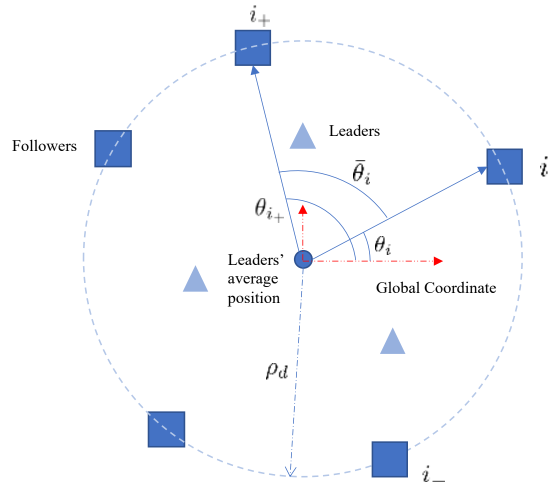

Fig. 1. Illustration for the enclosing problem

$$\dot{\boldsymbol{p}}_i^L = \boldsymbol{v}_i, \forall i \in \mathcal{V}_T \tag{1}$$

$$\dot{\boldsymbol{p}}_i^F = \boldsymbol{u}_i, \forall i \in \mathcal{V}_F \tag{2}$$

where $\boldsymbol{p}_i^L, \boldsymbol{p}_i^T$ and $\boldsymbol{v}_i, \boldsymbol{u}_i$ are the positions and velocities of leaders and followers, respectively. We assume that these agents are moving on 2D plane, i.e. $\boldsymbol{u}_i \in \mathbb{R}^2 = [u_i^x, u_i^y]^T$. In the following, we consider the case where the control of follower is constrained by a maximum output $\bar{\boldsymbol{V}}$, that is:

$$\boldsymbol{u}_i = \begin{cases} [\bar{V}, w_i^y]^T, & \text{if } w_i^x > \bar{V} \\ [w_i^x, \bar{V},]^T, & \text{if } w_i^y > \bar{V} \\ \boldsymbol{w}_i, & \text{if } \underline{V} < w_i^x, w_i^x < \bar{V} \\ [\underline{V}, w_i^y]^T, & \text{if } w_i^x < \underline{V} \\ [w_i^x, \underline{V},]^T, & \text{if } w_i^y < \underline{V} \end{cases} \tag{3}$$

Let define $\Delta\boldsymbol{u}_i = \boldsymbol{u}_i - \boldsymbol{w}_i$ be the saturation value that each agent can monitor by themselves. Without loss of generality, we assume that the saturation is upper bounded.

Assumption 1. *Assume that the value of control saturation* Δu *is upper-bounded, i.e. there exists a positive constant* $\bar{U}$ *such that* $|\Delta u| \leq \bar{U}$.

The enclosing problem and the desired enclosing formation has been depicted in Fig. 1. Let define the geometrical center of leaders as the Leaders' Average Position (LAP) by

$$\bar{\boldsymbol{p}}^T = \frac{1}{m} \sum_{\forall i \in \mathcal{V}_F} \boldsymbol{p}_i^T \tag{4}$$

Then the relative position between follower $i \in \mathcal{V}_F$ and the target can be denoted by:

$$\Delta\boldsymbol{p}_i = \boldsymbol{p}_i - \bar{\boldsymbol{p}}^{\boldsymbol{T}} = [\Delta p_i^x, \Delta p_i^y]^T$$

And the relative angle of follower i with respect to LAP in the global frame is,

$$\theta_i = atan2(\frac{\Delta p_i^y}{\Delta p_i^x})$$

Define the unit vector along the line passing through the LAP and follower i as

$$\boldsymbol{\varphi}_i = \frac{\Delta \boldsymbol{p}_i}{\rho} = [cos(\theta_i), sin(\theta_i)]^T$$

Moreover, denote the unit normal vector to $\boldsymbol{\varphi}_i$ by

$$\boldsymbol{\varphi}_{i\perp} = [-sin(\theta_i), cos(\theta_i)]^T$$

Therefore the included angle from agent i to j is defined by:

$$\bar{\theta}_i = \theta_{i_+} - \theta_i + \varsigma_i \tag{5}$$

where

$$\varsigma_i = \begin{cases} 0, \text{if } \theta_{i_+} - \theta_i \geq 0 \\ 2\pi, \text{if } \theta_{i_+} - \theta_i < 0 \end{cases} \tag{6}$$

Given the aforementioned definitions, the desired enclosing formation is encoded by a triple tuple $\mathbb{P}_d = \{w_d, \rho_d, \boldsymbol{a}\}$, which are respectively the desired rotating angular velocity w_d, the desired relative distance between the followers and LAP ρ_d and the set of desired included angle between any two consecutive followers $\boldsymbol{a} = [a_1, a_2, \cdots, a_n]^T$. Given a predefined enclosing formation, it is called admissible if and only if

$$\rho_d > max(||\boldsymbol{p}_i^t - \bar{\boldsymbol{p}}^T||), a_i > 0 \text{ and } \sum_{i \in \mathcal{V}_F} \varsigma_i = 2\pi.$$

One key to the enclosing problem is to obtain the exact LAP. In contrast to the definition in (4), the followers should distributively compute this value based on the information locally collected. A finite-time estimator for computing the LAP in each agent is designed as following:

$$\begin{aligned} \boldsymbol{r}_i &= \boldsymbol{\Phi}_i + \tilde{\boldsymbol{p}}_i, \qquad \tilde{\boldsymbol{p}}_i = \frac{n}{m} \sum_{j \in \boldsymbol{N}_i^T} \frac{1}{|\boldsymbol{N}_j^F|} \boldsymbol{p}_j. \\ \dot{\boldsymbol{\Phi}}_i &= \beta \sum_{j \in \boldsymbol{N}_i} \frac{\boldsymbol{r}_j - \boldsymbol{r}_i}{|\boldsymbol{r}_j - \boldsymbol{r}_i|} \end{aligned} \tag{7}$$

It has been proved to be stable in finite time in [11]–[13] as long as the communication graph among followers is connected and the estimator gain has

$$\beta > n(n-1)|\dot{\boldsymbol{r}}_i|, \forall i \in \mathcal{V}_F$$

In the following, we directly apply this method for estimating the LAP $\bar{\boldsymbol{p}}$ using $\boldsymbol{r}_i, \forall i \in \mathcal{V}_F$ for all followers. Therefore, the variables $\Delta \boldsymbol{p}_i, \theta_i, \boldsymbol{\phi}_i, \boldsymbol{\phi}_{\perp,i}$ used in the following paper are all computed based on the estimated LAP $\boldsymbol{r}_i$. The left problem is to design proper enclosing controllers such that a predefined spacing pattern $\mathbb{P}_d$ can always be achieved in finite time while considering the control input saturation.

IV. Enclosing controller design

Let define the error of relative distance and included angle as:

$$\begin{aligned} z_i &= \rho_i - \rho_d \\ \delta_i &= \hat{\theta} - 1, \quad \hat{\theta} = \frac{\bar{\theta}_i}{a_i}. \end{aligned} \tag{8}$$

It's easy to verify that $\sum_{i=1}^n a_i \delta_i = \sum_{i=1}^n \bar{\theta}_i - \sum_{i=1}^n a_i = 2\pi - 2\pi = 0$, i.e. $\boldsymbol{a}^T \boldsymbol{\delta} = 0$. The derivative of relative distance error has :

$$\begin{aligned} \dot{z}_i &= \dot{\rho}_i \\ &= \frac{d}{dt} \sqrt{(\Delta p_i^x)^2 + (\Delta p_i^y)^2} \\ &= \frac{1}{2\rho_i} \Delta \dot{p}_i^x \Delta p_i^x + \Delta \dot{p}_i^y \Delta p_i^y \\ &= \frac{1}{2\rho_i} \Delta \dot{\boldsymbol{p}}_i^T \Delta \boldsymbol{p}_i = \frac{1}{2\rho_i} \Delta \boldsymbol{p}_i^T \Delta \dot{\boldsymbol{p}}_i = \frac{1}{2} \boldsymbol{\varphi}_i^T \Delta \dot{\boldsymbol{p}}_i \end{aligned} \tag{9}$$

Regarding the included angle error, one has:

$$\dot{\delta}_i = \frac{\dot{\bar{\theta}}_i}{a_i}, \quad \dot{\bar{\theta}}_i = \dot{\theta}_{i_+} - \dot{\theta}_i \tag{10}$$

According to the definition of the included angle, we have:

$$\begin{aligned} \dot{\theta}_i &= \frac{1}{1 + (\frac{\Delta p_i^y}{\Delta p_i^y})^2} \frac{d}{dt} (\frac{\Delta p_i^y}{\Delta p_i^y}) \\ &= \frac{(\Delta p_i^x)^2}{\rho_i^2} \frac{\Delta \dot{p}_i^y \Delta p_i^x - \Delta \dot{p}_i^x \Delta p_i^y}{(\Delta p_i^x)^2} \\ &= \frac{1}{\rho_i} [-\Delta p_i^y, \Delta p_i^x][-\Delta \dot{p}_i^x, \Delta \dot{p}_i^y]^T \frac{1}{\rho_i} \\ &= \boldsymbol{\varphi}_{i\perp}^T \Delta \dot{\boldsymbol{p}}_i \frac{1}{\rho_i} \end{aligned} \tag{11}$$

Let $\Delta \boldsymbol{p} = [\Delta \boldsymbol{p}_1, \Delta \boldsymbol{p}_2, \cdots, \Delta \boldsymbol{p}_n]^T, \boldsymbol{z} = [z_1 z_2, \cdots, z_n]^T, \boldsymbol{\delta} = [\delta_1, \delta_2, \cdots, \delta_n]^T$ be the compact vector of these variables respectively. Then we can conclude the dynamics of both errors as:

$$\dot{\boldsymbol{z}} = \frac{1}{2} diag([\boldsymbol{\varphi}_i^T]) \Delta \dot{\boldsymbol{p}} = \frac{1}{2} \boldsymbol{\phi} \Delta \dot{\boldsymbol{p}} \tag{12}$$

$$\dot{\boldsymbol{\delta}} = -\boldsymbol{D}^{-1} \boldsymbol{\Lambda}^{-1} \boldsymbol{L}_\theta \boldsymbol{\phi}_\perp \Delta \dot{\boldsymbol{p}} \tag{13}$$

where $\boldsymbol{D} = \in \mathbb{R}^{n \times n} = [\rho_i], \boldsymbol{L}_\theta \in \mathbb{R}^{n \times n} = [L_{ij}]$ and $L_{i,i} = 1, L_{i,i_+} = -1$ and other elements are all zeros. The newly introduced matrices are defined as $\boldsymbol{\Lambda} = diag(\boldsymbol{a}) \in \mathbb{R}^{n \times n}, \boldsymbol{\phi} = diag([\boldsymbol{\varphi}_i^T]) \in \mathbb{R}^{n \times 2n}$, and $\boldsymbol{\phi}_\perp = diag([\boldsymbol{\varphi}_{i\perp}^T]) \in \mathbb{R}^{n \times 2n}$.

A. Sliding mode for input saturation

Define the sliding variables:

$$\begin{cases} e_{z,i} = z_i - \lambda_{z,i} \\ e_{\delta,i} = \delta_i - \frac{1}{a_i} \sum_{i=1}^n l_{i,j} (\lambda_{\delta,i} - \lambda_{\delta,j}) \end{cases} \tag{14}$$

where the latent variables $\lambda_{z,i}$ and $\lambda_{\delta,i}$ will be designed later and $a_i, \forall i \in \mathcal{V}_F$ is the desired included angle as defined in the spacing pattern $\mathbb{P}_d$. Define the compact form of the sliding variables as $\boldsymbol{e}_z = [e_{z,1}, e_{z,2}, \cdots, e_{z,n}]^T, \boldsymbol{e}_\lambda =$

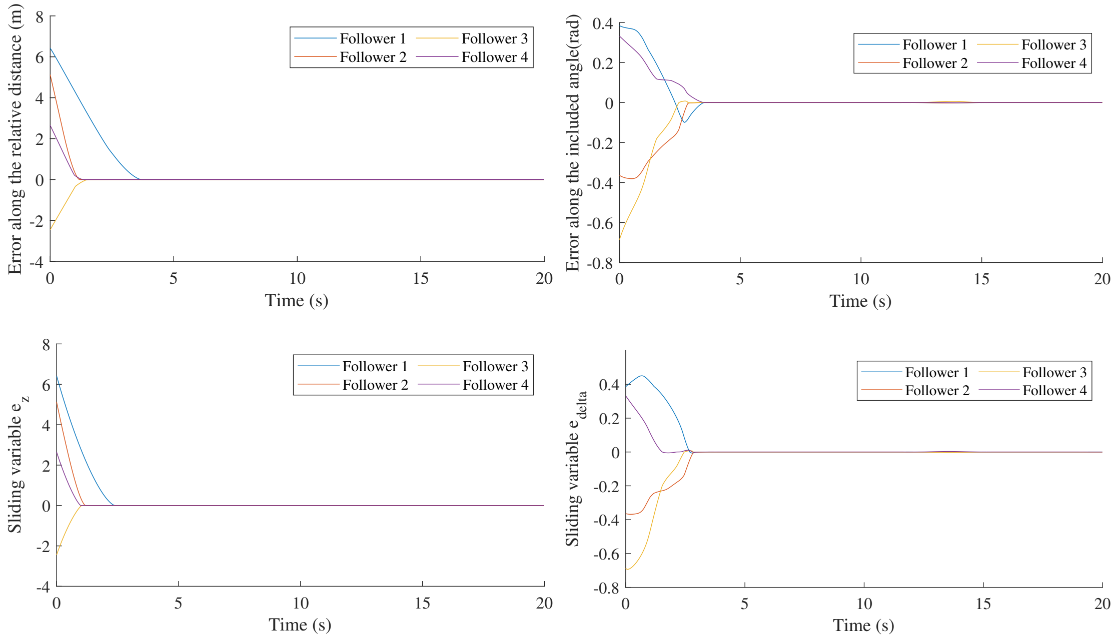

Fig. 2. The evolution of simulation results. The first two columns are the errors of the enclosing formation z_i, δ_i over $20s$ respectively. The second columns are the sliding variables e_z, e_δ over $20s$ respectively.

$[e_{\lambda,1}, e_{\lambda,1}, \cdots, e_{\lambda,n}]^T$, then taking the derivative of the sliding variable $\boldsymbol{e}_z$ yielding

$$\begin{aligned} \dot{\boldsymbol{e}}_z &= \dot{\boldsymbol{z}} - \dot{\boldsymbol{\lambda}}_z \\ &= \frac{1}{2}\boldsymbol{\phi}\Delta\dot{\boldsymbol{p}} - \dot{\boldsymbol{\lambda}}_z \end{aligned} \tag{15}$$

Given that $\Delta\dot{\boldsymbol{p}} = \boldsymbol{u} - \boldsymbol{v}$, then

$$\dot{\boldsymbol{e}}_z = \frac{1}{2}\boldsymbol{\phi}\left[\boldsymbol{u} - \boldsymbol{v}\right] - \dot{\boldsymbol{\lambda}}_z \tag{16}$$

where $\boldsymbol{v}$ is the velocity of LAP. As a result, if the latent variable $\boldsymbol{\lambda_z}$ has the following dynamic:

$$\dot{\boldsymbol{\lambda}}_z = -k_1\boldsymbol{\lambda}_z^{\frac{1}{\alpha}} - k_2 sign(\boldsymbol{\lambda}_z) + \frac{1}{2}\boldsymbol{\phi}\Delta\boldsymbol{u} \tag{17}$$

Then

$$\begin{aligned} \dot{\boldsymbol{e}}_z &= \frac{1}{2}\boldsymbol{\phi}\left[\boldsymbol{u} - \boldsymbol{v} - \Delta\boldsymbol{u}\right] + k_1\boldsymbol{\lambda}_z^{\frac{1}{\alpha}} + k_2 sign(\boldsymbol{\lambda}_z) \\ &= \frac{1}{2}\boldsymbol{\phi}\left[\boldsymbol{w} - \boldsymbol{v}\right] + k_1\boldsymbol{\lambda}_z^{\frac{1}{\alpha}} + k_2 sign(\boldsymbol{\lambda}_z) \end{aligned} \tag{18}$$

In addition, let the control command $\boldsymbol{w}$ be composed of two parts:

$$\boldsymbol{w} = \boldsymbol{\phi}^T\boldsymbol{w}_z + \boldsymbol{\phi}_\perp^T\boldsymbol{w}_\delta \tag{19}$$

where $\boldsymbol{w}_z, \boldsymbol{w}_\delta \in \mathbb{R}^{n\times n}$. As a result,

$$\dot{\boldsymbol{e}}_z = \frac{1}{2}\boldsymbol{w}_z - \frac{1}{2}\boldsymbol{\phi}\boldsymbol{v} + k_1\boldsymbol{\lambda}_z^{\frac{1}{\alpha}} + k_2 sign(\boldsymbol{\lambda}_z) \tag{20}$$

If $\dot{\boldsymbol{e}}_z = -\frac{1}{2}\boldsymbol{e}^{\frac{1}{\alpha}}$ and α is an odd positive, then the control input $\boldsymbol{w}_z$ can be designed as:

$$\boldsymbol{w}_z = -\boldsymbol{e}^{\frac{1}{\alpha}} + \boldsymbol{\phi}\dot{\boldsymbol{r}} - 2k_1\boldsymbol{\lambda}_z^{\frac{1}{\alpha}} - 2k_2 sign(\boldsymbol{\lambda}_z) \tag{21}$$

Where $\dot{\boldsymbol{r}}$ is the derivative of estimator output in (7). Considering the Lyapunouv function $V_1 = \frac{1}{2}\boldsymbol{e}_z^T\boldsymbol{e}_z = \frac{1}{2}\sum_{i=1}^n e_{z,i}^2$. According to lemma 1, its derivative has:

$$\begin{aligned} \dot{V}_1 &= -\sum_{i=1}^n e_{z,i}\dot{e}_{z,i} = -\sum_{i=1}^n e_{z,i}^{1+\frac{1}{\alpha}} \\ &= -\sum_{i=1}^n \left(e_{z,i}^2\right)^{\frac{\alpha+1}{2\alpha}} \leq -\left(\sum_{i=1}^n e_{z,i}^2\right)^{\frac{\alpha+1}{2\alpha}} = -V_1^{\frac{\alpha+1}{2\alpha}} \end{aligned} \tag{22}$$

Therefore, the sliding variable $\boldsymbol{e}_z$ will converge to zeros in finite time $T_1 \leq \frac{2\alpha}{\alpha-1}V(0)^{\frac{\alpha-1}{2\alpha}}$.

Regarding the sliding variable $\boldsymbol{e}_\delta$, its derivative has:

$$\begin{aligned} \dot{\boldsymbol{e}}_\delta &= \dot{\boldsymbol{\delta}} - \boldsymbol{\Lambda}^{-1}\boldsymbol{L}_\theta\dot{\boldsymbol{\lambda}}_\delta \\ &= -\boldsymbol{\Lambda}^{-1}\boldsymbol{L}_\theta(\boldsymbol{\phi}_\perp\Delta\dot{\boldsymbol{p}} + \boldsymbol{D}\dot{\boldsymbol{\lambda}}_\delta) \end{aligned} \tag{23}$$

Then the latent variable $\boldsymbol{\lambda}_\delta$ can be designed as:

$$\dot{\boldsymbol{\lambda}}_\delta = -\boldsymbol{D}^{-1}(k_3\boldsymbol{\lambda}_\delta^{\frac{1}{\alpha}} + k_4 sign(\boldsymbol{\lambda}_\delta) + \boldsymbol{\phi}_\perp\Delta\boldsymbol{u}) \tag{24}$$

Then

$$\begin{aligned} \dot{\boldsymbol{e}}_\delta &= -\boldsymbol{D}^{-1}\boldsymbol{\Lambda}^{-1}\boldsymbol{L}_\theta\left[\boldsymbol{\phi}_\perp(\boldsymbol{u} - \boldsymbol{v} - \Delta\boldsymbol{u}) - k_3\boldsymbol{\lambda}^{\frac{1}{\alpha}} - k_4 sign(\boldsymbol{\lambda}_\delta)\right] \\ &= -\boldsymbol{D}^{-1}\boldsymbol{\Lambda}^{-1}\boldsymbol{L}_\theta\left[\boldsymbol{w}_\delta - \boldsymbol{\phi}_\perp\boldsymbol{v} - k_3\boldsymbol{\lambda}^{\frac{1}{\alpha}} - k_4 sign(\boldsymbol{\lambda}_\delta)\right] \end{aligned} \tag{25}$$

Let

$$\boldsymbol{w}_\delta = \boldsymbol{\phi}_\perp\dot{\boldsymbol{r}} + k_3\boldsymbol{\lambda}^{\frac{1}{\alpha}} + k_4 sign(\boldsymbol{\lambda}_\delta) + \boldsymbol{e}_\delta^{\frac{1}{\alpha}} + w_d \tag{26}$$

where w_d is the desired rotating velocity from the predefined spacing pattern $\mathbb{P}_d$, then

$$\dot{\boldsymbol{e}}_\delta = -\boldsymbol{D}^{-1}\boldsymbol{\Lambda}^{-1}\boldsymbol{L}_\theta\boldsymbol{e}_\delta^{\frac{1}{\alpha}}$$

.

Lemma 2. *Let $\alpha > 1$ be a positive odd, then the origin of*

$$\dot{\boldsymbol{e}}_\delta = -\boldsymbol{D}^{-1}\boldsymbol{\Lambda}^{-1}\boldsymbol{L}_\theta \boldsymbol{e}_\delta^{\frac{1}{\alpha}}$$

is globally asymptotically stable.

Proof. Considering the Lyapunouv function $V_2 = \frac{\alpha+1}{\alpha}\boldsymbol{e}_\delta^T \boldsymbol{e}_\delta^{\frac{1}{\alpha}} = \frac{\alpha+1}{\alpha}\sum_{i=1}^n e_{\delta,i}^{1+\frac{1}{\alpha}}$, then its first derivative has:

$$\begin{aligned}\dot{V}_2 &= \sum_{i=1}^n e_{\delta,i}^{\frac{1}{\alpha}}\dot{e}_{\delta,i} = -\sum_{i=1}^n e_{\delta,i}^{\frac{1}{\alpha}}\frac{1}{a_i\rho_i}\sum_{j=1}^n l_{ij}(e_{\delta,i}^{\frac{1}{\alpha}} - e_{\delta,j}^{\frac{1}{\alpha}}) \\ &= -\frac{1}{2}\sum_{i=1}^n \frac{1}{a_i\rho_i}\sum_{j=1}^n l_{ij}(e_{\delta,i}^{\frac{1}{\alpha}} - e_{\delta,j}^{\frac{1}{\alpha}})^2 \le 0\end{aligned} \tag{27}$$

The equality holds if and only if $\boldsymbol{e}_\delta = e\boldsymbol{1}$, where e is a constant. However, since the error vector has the relationship of $\boldsymbol{e}_\delta = \boldsymbol{\delta} - \boldsymbol{\Lambda}^{-1}\boldsymbol{L}_\theta\boldsymbol{\lambda}$, and $\boldsymbol{1}^T\boldsymbol{L}_\theta = \boldsymbol{0}$, then we can get:

$$\begin{aligned}\boldsymbol{a}^T\boldsymbol{e}_\delta &= \boldsymbol{a}^T\boldsymbol{\delta} - \boldsymbol{a}^T\boldsymbol{\Lambda}^{-1}\boldsymbol{L}_\theta\boldsymbol{\lambda} \\ &= \boldsymbol{a}^T\boldsymbol{\delta} - \boldsymbol{1}^T\boldsymbol{L}_\theta\boldsymbol{\lambda} \\ &= 0\end{aligned} \tag{28}$$

As a result, the only solution for the equality is $e = 0$. Thus, we have $\dot{V}_2 \le 0$ with $\dot{V}_2 = 0$ if and only if $\boldsymbol{e}_\delta = \boldsymbol{0}$. Therefore, the origin is globally asymptotically stable. □

Lemma 3. *Define the vector field $\boldsymbol{H}(\boldsymbol{e}_\delta) = -\boldsymbol{D}^{-1}\boldsymbol{\Lambda}^{-1}\boldsymbol{L}_\theta\boldsymbol{e}_\delta^{\frac{1}{\alpha}}$, then it is homogeneous of degree $\frac{1}{\alpha}-1$ with respect to the standard dilation.*

Proof. Firstly, we show that $e_{\delta,i}^{\frac{1}{\alpha}}$ is homogeneous of degree $\frac{1}{\alpha}$ with respect to the standard dilation, i.e. for a $\lambda > 0$:

$$(\lambda e_{\delta,i})^{\frac{1}{\alpha}} = \lambda^{\frac{1}{\alpha}} e_{\delta,i}^{\frac{1}{\alpha}}$$

Then, let's denote the $\boldsymbol{i}$-th component of the vector field $\boldsymbol{H}(\boldsymbol{e}_\delta)$ as $\boldsymbol{H}_i(\boldsymbol{e}_\delta) = -\frac{1}{a_i\rho_i}\sum_{j=1}^n l_{ij}(e_{\delta,i}^{\frac{1}{\alpha}} - e_{\delta,j}^{\frac{1}{\alpha}})$, whereby we can get:

$$\begin{aligned}\boldsymbol{H}_i(\lambda\boldsymbol{e}_\delta) &= -\frac{1}{a_i\rho_i}\sum_{j=1}^n l_{ij}\left[(\lambda e_{\delta,i})^{\frac{1}{\alpha}} - (\lambda e_{\delta,j})^{\frac{1}{\alpha}}\right] \\ &= -\lambda^{\frac{1}{\alpha}}\frac{1}{a_i\rho_i}\sum_{j=1}^n l_{ij}\left(e_{\delta,i}^{\frac{1}{\alpha}} - e_{\delta,j}^{\frac{1}{\alpha}}\right) \\ &= \lambda^{\frac{1}{\alpha}}\boldsymbol{H}_i(\boldsymbol{e}_\delta)\end{aligned}$$

Therefore, we say that the vector filed is homogeneous of degree $\frac{1}{\alpha}-1$ with respect to the standard dilation. □

TABLE I
PARAMETERS FOR THE DESIGNED CONTROLLER

k_1	k_2	k_3	k_4	α	β
1.5	4	1.5	4	3	10

Therefore, according to theorem 2, the sliding variable $\boldsymbol{e}_\delta$ will also converge to zero in finite time. As a result, we can conclude the following theorem,

Theorem 3. *Given the latent variables defined in (17) and (24), according to the compact form of control commands in (19) with sub-terms (21) and (26), where its distributed control command for each agent can be rewritten as:*

$$\begin{cases}\boldsymbol{w}_i = \boldsymbol{\varphi}_i^T\boldsymbol{w}_{z,i} + \boldsymbol{\varphi}_{\perp,i}^T\boldsymbol{w}_{\delta,i} \\ \boldsymbol{w}_{z,i} = -\boldsymbol{e}_i^{\frac{1}{\alpha}} + \boldsymbol{\varphi}_i^T\dot{\boldsymbol{r}}_i - 2k_1\boldsymbol{\lambda}_z^{\frac{1}{\alpha}} - 2k_2 sign(\boldsymbol{\lambda}_{z,i}) \\ \boldsymbol{w}_{\delta,i} = \boldsymbol{\varphi}_{\perp,i}^T\dot{\boldsymbol{r}}_i + k_3\boldsymbol{\lambda}_{\delta,i}^{\frac{1}{\alpha}} + k_4 sign(\boldsymbol{\lambda}_{\delta,i}) + \boldsymbol{e}_{\delta,i}^{\frac{1}{\alpha}} + w_d\end{cases} \tag{29}$$

then the error of spacing pattern z_i, δ_i defined in (8) can both converge to zero in finite time if $k_1 > 0, k_3 > 0, k_2 > \frac{\bar{U}}{2}, k_4 > \bar{U}$ and $\alpha > 1$ be a positive odd, i.e. the desired enclosing formation $\mathbb{P}_d$ can be achieved.

Proof. From the previous analysis, the sliding variables $\boldsymbol{e}_z, \boldsymbol{e}_\delta$ will both converge to zeros in finite time. Now let's consider the convergence of the latent variable $\lambda_{z,i}$ and $\lambda_{\delta,i}$. Taking $\lambda_{z,i}$ as an example, let define the Lyapunouv candidate function $V_3 = \frac{1}{2}\boldsymbol{\lambda}_z^T\boldsymbol{\lambda}_z$. Consider the derivative in (17), then we have,

$$\begin{aligned}\dot{V}_3 &= -k_1\boldsymbol{\lambda}_z^{1+\frac{1}{\alpha}} - k_2|\boldsymbol{\lambda}_z| + \frac{1}{2}\boldsymbol{\lambda}_z^T\boldsymbol{\phi}\Delta\boldsymbol{u} \\ &\le -k_1\boldsymbol{\lambda}_z^{1+\frac{1}{\alpha}} - (k_2 - \frac{1}{2}\bar{U})|\boldsymbol{\lambda}_z|\end{aligned} \tag{30}$$

Then if $k_2 \ge \frac{1}{2}\bar{U}$, $\dot{V}_3 \le -k_1\lambda_z^{1+\frac{1}{\alpha}} = -k_1 V_3^{\frac{1+\alpha}{2\alpha}}$. Therefore, we know that $\lambda_{z,i} = 0, \forall i \in \mathcal{V}_F$ in finite time. Similarly, we can also derive that $\lambda_{\delta,i} = 0, \forall i \in \mathcal{V}_F$ is finite-time stable if $k_4 > \bar{U}$. Then recalling the sliding variables in (8), we prove that the error of spacing pattern $z_i, \delta_i, \forall i \in \mathcal{V}_F$ is finite-time stable.

This completes the proof of theorem 3 □

TABLE II
INITIAL POSITIONS FOR THE SIMULATION CASE

Nodes	Followers				Leaders	
	F_1	F_2	F_3	F_4	L_1	L_2
Pos(m)	[5; −10]	[9;6]	[1;3]	[−1;8]	[0;0]	[1;1]

V. SIMULATION RESULTS

The simulation considers two moving leaders whose moving velocities are respectively,

$$\begin{cases} v_1^x = 0.5 \\ v_1^y = 0.5 \end{cases} \quad \begin{cases} v_2^x = 3sin(\frac{t}{2} + \frac{\pi}{3}) + 0.5 \\ v_2^y = 0.5 \end{cases} \tag{31}$$

We are deploying four followers applying the controller as defined in (29) in the same moving plane to encircle around these leaders. The initial positions are given in the Table II. The desired spacing pattern is defined as $\mathbb{P}_d := \{w_d = 1, \rho_d = 5, \boldsymbol{a} = [\frac{2\pi}{5}, \frac{2\pi}{5}, \frac{2\pi}{5}, \frac{4\pi}{5}]\}$. The parameters in both estimator and controller are designed in Tab. I. Moreover, the real control inputs of the followers are upper bounded by $\bar{V} = 2.5, \underline{V} = -2.5$.

The simulation results are shown in Fig. 2,3 and 4. It can be shown from these figures that both the defined sliding variables $e_{z,i}, e_{\delta,i}$ and the spacing pattern error δ_i, z_i are finite-time convergent. Besides, the desired spacing pattern can be derived and maintained using the designed controller in (29) and

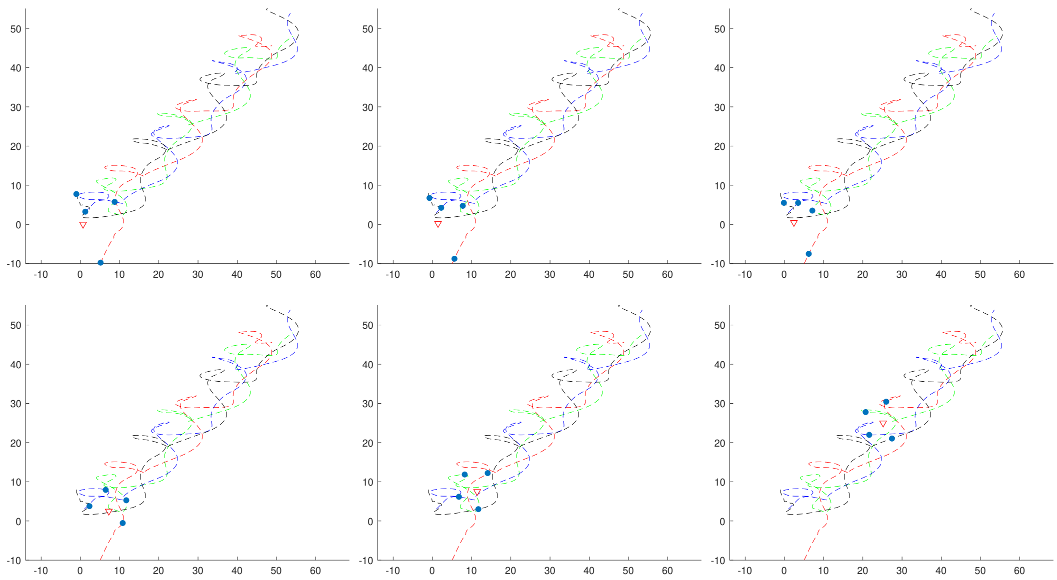

Fig. 3. The evolution of enclosing formation over 50s. According to the left-right and up-down sequence, these formations are captured at 0.1s, 0.5s, 1s, 5s, 15s, 50s respectively. The blue dots are the followers and red triangle represents the LAP

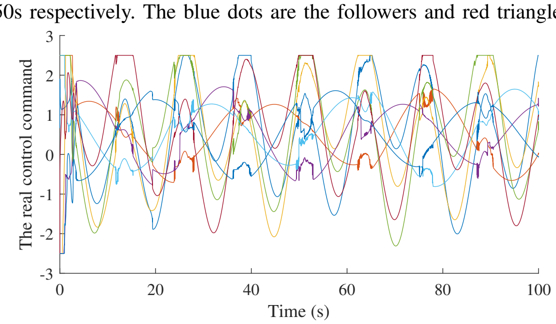

Fig. 4. The saturated control commands of all four followers

estimator (7), which validates the effectiveness and correctness of the proposed methods. Moreover, the saturation during the control process is illustrated in Fig 4.

VI. Conclusion

This paper investigates the problem of finite-time enclosing problem for multiple mobile targets using a MAS with control input saturation. Tow novel finite-time sliding modes are proposed for the saturation encountered in the enclosing problem. By monitoring the difference between real executed control and the desired control, the saturation is compensated. Future work may focus on remove the assumption of upper boundary of the control saturation and extended such enclosing problem into more realistic scenarios.

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